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ERROR BOUNDS FOR THE LANCHESTER EQUATIONS WITH VARIABLE COEFFICIENTS

by

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Previous error bounds for the classical Liouville-Green solutions to second order ordinary differential equaitons are sharpened. Applications are made to the Lanchester model for combat between two homogeneous forces.

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We are interested in solving the coupled set of equations

$$\frac{dx}{dt} = -a(t)y$$
$$\frac{dy}{dt} = -b(t)x \tag{1}$$

For the problems we are interested in, these equations do not reduce to any known special functions. These equations arise in the study of combat models where x and y are opposing forces and $a(t)$ and $b(t)$ are the attrition rate coefficients related to the effectiveness of weapons. Lanchester [1] first looked at these types of models for aircraft battles, using constant coefficients (in which case the solution is in terms of exponentials) and so these types of equations are referred to as Lanchester equations. There are also special cases of the coefficients for which the solutions to (1) are expressible in terms of the generalized Airy functions [2].

We are interested in problems where the coefficients $a(t)$ and $b(t)$ are positive and continuous. In that case we expect that we can find solutions which are linear combinations of monotonic, exponential like functions. If we have such a set, then the analysis of the forces necessary to guarantee a win for one side, say x, is very much simplified. So we are interested in determining exponential like solutions for (1), and error bounds for the solutions.

We convert (1) into a single second order differential equations for $x(t)$

$$\frac{d^2x}{dt^2} - \left\{a^{-1}\frac{da}{dt}\right\}\frac{dx}{dt} - a(t)b(t)x = 0$$
$$x(0) = x_0 \; , \; a^{-1}(0)\frac{dx}{dt}(0) = -y_0 \tag{2}$$

Equation (2) is still awkward. Let us define two useful quantities, the intensity of combat $I(t)$ and relative fire effectiveness $R(t)$,

$$I(t) \equiv \sqrt{a(t)b(t)} \tag{3}$$

$$R(t) \equiv \sqrt{\frac{a(t)}{b(t)}} \ . \tag{4}$$

A change of independent variables will simplify (2). Let

$$\tau \equiv \int_0^t I(s)ds \ , \tag{5}$$

then (2) becomes

$$\frac{d^2x}{d\tau^2} - R^{-1}\frac{dR}{d\tau}\frac{dx}{d\tau} - x = 0 \ . \tag{6}$$

We now reduce (6) to the standard Liouville-Green form by the usual change of dependent variable

$$W = \frac{x}{\sqrt{R}} \ . \tag{7}$$

So (6) becomes

$$\frac{d^2W}{d\tau^2} - \left[1 + \sqrt{R}\frac{d^2}{d\tau^2}(1/\sqrt{R})\right]W = 0 \ . \tag{8}$$

Then the standard estimates for (8) say that there are two solutions to (8)

$$W = c_1e^{-\tau} + c_2e^{\tau} \ ,$$

i.e. there are two solutions to (2)

$$x(t) = \sqrt[4]{\frac{b(t)}{a(t)}}\left\{c_1e^{\int_0^t I(s)ds} + c_2e^{-\int_0^t I(s)ds}\right\} . \tag{9}$$

What is the error in using (9)? A number of error estimates are availabl [3], [4]. It turns out that they are not quite good enough for our purposes. The estimates depend upon whether one is interested in the dominant solution or in the subdominant solution.

Looking first at the dominant solution, e^{τ} , we take

$$w_1 = e^{\tau}[1 + h_1(\tau)] \quad . \tag{10}$$

Then the error $h_1(\tau)$ satisfies

$$h_1'' + 2h_1' = \psi(1 + h_1) \tag{11}$$

where

$$\psi \equiv \sqrt{R}\,\frac{d^2}{d\tau^2}\,(1/\sqrt{R}) \quad . \tag{12}$$

Converting (11) to an integral equation with zero initial conditions we have

$$h_1(\tau) = \frac{1}{2}\int_0^{\tau}\left[1 - e^{-2(\tau-s)}\right]\psi(s)[1 + h_1(s)]ds \quad . \tag{13}$$

Our error estimates are based on Willet's generalization of Gronwall's lemma.

<u>Lemma 1</u>. Let

$$h(\tau) = \int_0^{\tau} K(s,\tau)\psi(s)[J(s) + h(s)]ds$$

and let

$$|K(s,\tau)| \le Q(s)P(\tau) \ ,$$

then

$$|h(\tau)| \le P(\tau)\int_0^{\tau} |J(s)|Q(s)|\psi(s)|\ \exp\left\{\int_s^{\tau} P(z)Q(z)|\psi(z)|dz\right\}ds \quad . \tag{14}$$

Proof: See Willet [5].

An immediate corollary is:

<u>Corollary</u>. Let $M(z) = Q(z)\max\{|J(z)|,P(z)\}$ then

$$|h(\tau)| \le P(\tau)\left[\exp\left\{\int_0^{\tau} M(s)|\psi(s)|ds\right\} - 1\right] . \tag{15}$$

<u>QED</u>

Proof: Integrate (14), using $M(z)$.

In the case of equation (13)

$$K(s,\tau) = \frac{1}{2}(1 - e^{-2(\tau - s)}) \le \frac{1}{2}(1 - e^{-2\tau})$$

$$J(s) = 1 \quad .$$

Thus we can take $P(\tau) = 1 - e^{-2\tau}$ and $Q(s) = \frac{1}{2}$. Then lemma 1 gives

Theorem 1. The dominant solution to (8) satisfies

$$W_1 = e^{\tau}[1 + h_1(\tau)]$$

$$\leq e^{\tau}\left[1 + (1 - e^{-2\tau})\left(\exp\left\{\int_0^{\tau} \frac{1}{2} |\psi(s)| ds\right\} - 1\right)\right] \tag{16}$$

QED

As for the subdominant solution, we take

$$W_2 = e^{-\tau}[1 + h_2(\tau)]$$

and then h_2 satisfies

$$h_2'' - 2h_2' = \psi(1 + h_2) \tag{17}$$

which converts to

$$h_2(\tau) = \int_0^{\tau} \frac{1}{2}\left[e^{2(\tau-s)} - 1\right] \psi(s)[1 + h_2(s)]ds \tag{18}$$

Now we have

$$K(s,\tau) = \frac{1}{2} \quad e^{2(\tau-s)} - 1 \quad \leq \frac{1}{2}(e^{2\tau} - 1)e^{-2s} \quad .$$

Then we have

Theorem 2. The subdominant solution to (8) satisfies

$$W_2 = e^{-\tau}[1 + h_2(\tau)]$$

$$\leq e^{-\tau}\left[1 + (e^{2\tau} - 1)\int_0^{\tau} e^{-2s} \left|\frac{\psi(s)}{2}\right| \exp \int_s^{\tau} (1 - e^{-2z}) \left|\frac{\psi(z)}{2}\right| dz \, ds\right] . \tag{19}$$

QED

Observe that we did not use the corollary on (19).

We can see that our estimates depend very strongly on ψ , given by (12).

For the Lanchester equations (1), the coefficients $a(t)$ and $b(t)$, for many applications, can be expressed by

$$\begin{aligned} a(t) &= k_a(t + c)^{\mu} \\ b(t) &= k_b(t + c + a)^{\upsilon} \end{aligned} \tag{20}$$

where c is the "starting" parameter, and A is an "off-set" parameter (so y can have a different firing range than x). These are referred to as the "power rate" coefficients. For future use, let

$$B = \sqrt{k_a k_b}$$

$$\delta = \frac{\mu + \upsilon + 2}{2} .$$

We first consider the case of no offset, i.e. $A = 0$. Then (see (5))

$$\tau = B \int_0^t (t + c)^{\frac{\mu+\upsilon}{2}} dt = B\left[\frac{(t + c)^\delta}{\delta} - \frac{c^\delta}{\delta}\right] \tag{21}$$

and

$$\psi = - \frac{\upsilon - \mu}{16} \frac{(\upsilon + 3\mu + 4)}{(\delta\tau + c^\delta)^2} . \tag{22}$$

We see that for $-1 < \mu < \upsilon$ that ψ is negative, and

$$\left[\exp \int_0^\tau \left|\frac{\psi(s)}{2}\right| ds\right] - 1$$

$$= -1 + \exp \left\{\left(\frac{\upsilon - \mu}{32\delta B}\right)\left(\frac{\upsilon + 3\mu + 4}{A^\delta}\right)\right\} \exp \left\{-\left(\frac{\upsilon - \mu}{32\delta B}\right)\left(\frac{\upsilon + 3\mu + 4}{A^\delta + \delta\tau}\right)\right\} .$$

Let

$$\gamma \equiv \frac{(\upsilon - \mu)}{32\delta B} (\upsilon + 3\mu + 4) \tag{23}$$

$$\text{and} \qquad D_1 = \exp (\gamma/A^\delta) .$$

Then (16) becomes

$$w_1 \le e^\tau(D_1 \exp (-\gamma/(A^\delta + \delta\tau)) + e^{-2\tau} - D_1 e^{-2\tau} \exp (-\gamma/(A^\delta + \delta\tau))$$

as an error term for the dominant solution.

For the subdominant solution the corollary gives too crude an estimate. Even with using the lemma, the error estimate is not good. Integrating by parts we get

$$w_2 = e^{-\tau}[1 + h_2(\tau)]$$

$$\leq e^{-\tau}[1 + (e^{2\tau} - 1)\left(1 - \exp\left[\int_0^\tau (1 - e^{-2z})\left|\frac{\psi}{2}\right| dz\right]\right.$$

$$\left. + \int_0^\tau \left|\frac{\psi}{2}\right| \exp\left[\int_s^\tau (1 - e^{-2z})\left|\frac{\psi(z)}{2}\right| dz\right] ds\right).$$

Working this out we get

$$w_2 \leq e^{-\tau}\left[e^{2\tau} + 0\left(\exp\left(\frac{\gamma}{\tau\delta + A^\delta}\right)\right)\right] \tag{25}$$

This is not a very exciting bound, but we are unable to do any better, using the Willet result.

We can, however, alter the Willet result. The subdominant solution error also satisfies the equation

$$h_2 = \frac{1}{2}\int_\tau^\infty [1 - e^{-2(s - \tau)}]\psi(s)[1 + h_2(s)]ds \quad . \tag{26}$$

We now prove

Lemma 2. If $h(\tau)$ satisfies

$$h(\tau) \leq \int_\tau^\infty K(s, \tau)\psi(s)[J(s) + h(s)]ds \tag{27}$$

and

$$|K(s, \tau)| \leq P(\tau)Q(s) \tag{28}$$

where K, ψ, and J are all ≥ 0 , then

$$h(\tau) \leq P(\tau)\int_\tau^\infty \psi(s)Q(s)J(s)\exp\left[\int_\tau^s P(\sigma)Q(\sigma)\psi(\sigma)d\sigma\right]ds \quad . \tag{29}$$

Proof. Eq. (27) can be written, using (28)

$$\frac{h(\tau)}{P(\tau)} \leq \int_\tau^\infty Q(s)\psi(s)J(s)ds + \int_\tau^\infty Q(s)\psi(s)h(s)ds \quad .$$

Differentiating

$$\left(\frac{h}{P}\right)' \le -Q(\tau)\psi(\tau)J(\tau) - Q(\tau)\psi(\tau)h(\tau)\frac{P(\tau)}{P(\tau)} \quad .$$

Therefore

$$\left(\frac{h}{P}\right)' + Q\psi P\left(\frac{h}{P}\right) \le -Q\psi J \quad .$$

Integrating we get

$$\frac{h(\tau)}{P(\tau)} \le \int_\tau^\infty Q(s)\psi(s)J(s)\exp\left[\int_\tau^s P(\sigma)Q(\sigma)\psi(\sigma)d\sigma\right]ds$$

$$+ K\exp\left[\int_\tau^\infty P(\sigma)Q(\sigma)\psi(\sigma)d\sigma\right] .$$

But $h(\infty) = 0$, so $K = 0$. <u>QED</u>

For equation (26) we have

$$K(s,\tau) = \frac{1}{2}\left[1 - e^{-2(s - \tau)}\right] \le 1 \cdot \frac{1}{2} \quad .$$

Thus

$$h_2 \le \int_\tau^\infty \frac{\psi(s)}{2} \exp\left[\int_\tau^s \frac{\psi}{2}(\sigma)d\sigma\right]ds$$

$$= \exp\left\{\int_\tau^\infty \frac{\psi(\sigma)}{2}d\sigma\right\} - 1 \quad . \tag{30}$$

This is the same bound found by Olver [3].

For the case of the power attrition coefficients with no offset we have

$$w_2 \le e^{-\tau}\exp\left(\frac{\gamma}{A^\delta + \delta\tau}\right) \tag{31}$$

which is a fairly reasonable bound.

For the case where $\mu = \upsilon = 1$, the linear case, we can also get some explicit results with offset. That is

$$a(t) = k_a(t + c)$$

$$b(t) = k_b(t + c + A) \quad .$$

Then

$$\tau = \frac{BA^2}{8}\left\{\eta\sqrt{\eta^2 - 1} - \ln\left(\eta + \sqrt{\eta^2 - 1}\right)\right\} \tag{32}$$

where

$$\eta \equiv 1 + \frac{2(t + c)}{A} \quad .$$

Then

$$\frac{1}{2}\int_{\tau_0}^{\tau} \psi(\sigma)d\sigma$$

$$= \frac{1}{16BA^2}\left\{\frac{4(t + c)^3 - 6A(t + c)^2 - 12A^2(t + c) - 7A^3}{[(t + c)(t + c + A)]^{3/2}}\right.$$

$$\left. - \frac{4c^3 - 6Ac^2 - 12A^2c - 7A^3}{[C(C + A)]^{3/2}}\right\} \tag{33}$$

while

$$\frac{1}{2}\int_{\tau}^{\infty} \psi(\sigma)d\sigma$$

$$= \frac{1}{16BA^2}\left\{4 - \frac{4(t + c)^3 - 6A(t + c)^2 - 12A^2(t + c) - 7A^3}{[(t + c)(t + c + A)]^{3/2}}\right\} \quad . \tag{34}$$

Both integrals are monotone increasing.

For the general case

$$\frac{1}{2}\int_{\tau_0}^{\tau} \psi(s)ds$$

$$= \frac{1}{2\sqrt{k_a k_b}}\int_{\eta_0}^{\eta}\left\{\frac{(\mu + \frac{3}{4}\mu^2)(\eta^2 + A\eta + A^2/4) - (\upsilon + \upsilon^2/4)(\eta^2 - A\eta + A^2/4)}{(\eta - A/2)^{2 + \mu/2}\,(\eta + A/2)^{2 + \upsilon/2}}\right.$$

$$\left. - \frac{\frac{1}{2}\mu\upsilon}{(\eta - A/2)^{1 + \mu/2}\,(\eta + A/2)^{1 + \upsilon/2}}\right\} d\eta \tag{35}$$

where $\eta \equiv t + c + A/2$.

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